Chromatic Homotopy Theory

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Abstract

This is a note on chromatic homotopy theory, rewriting Chapters 1,2,3,7 of Ravenel's orange book.

Contents

1 Notations

For a topological space X, let $\pi_*^S(X) :=$ be its stable homotopy groups.

Let S be the sphere spectrum.

Let SG be the Moore spectrum of an abelian group G.

For spaces X, Y, let $[X, Y]_*^S := \mathrm{colim}[\Sigma^{i+*}X, \Sigma^i Y]$ be the stable homotopy groups of maps.

For a homology theory E_*, let $\overline{E}_*$ be the associated homology theory. By abuse of notation, we also denote the coefficient ring by $E_* := E_*(pt)$.

2 Rough Idea of Chromatic Homotopy Theory

In number theory, we have the Sullivan fracture square:

$$\begin{array}{ccc} \mathbb{Z} & \longrightarrow & \prod_{p:\, prime} \mathbb{Z}_p \\ \downarrow & & \downarrow \\ \mathbb{Q} & \longrightarrow & \mathbb{Q} \otimes \left(\prod_{p:\, prime} \mathbb{Z}_p\right) \end{array}$$

There is a similar fracture square in homotopy theory. First, we have the similar definition of localization:

Definition 2.1 (E-acyclic, E-local, E-localization)**.** Let E_* be a generalized homology theory. A spectrum X is **E_*-acyclic** if $E_*(X) = 0$. A space Y is **E_*-local** if $[X, Y]_* = 0$ for any E-acyclic X.

An **E_*-localization** of a spectrum X is a map $\eta\colon X \to L_E X$ such that $E_*(\eta)$ is an isomorphism.

Theorem 2.2 (Bousfield)**.** *Such $L_E X$ always exists and is functorial in X.*

Theorem 2.3. *For any spectrum X,*

$$\begin{array}{ccc} X & \longrightarrow & \prod_{p:\, prime} L_p X \\ \downarrow & & \downarrow \\ L_{\mathbb{Q}} X & \longrightarrow & L_{\mathbb{Q}}\left(\prod_{p:\, prime} L_p X\right) \end{array}$$

where $L_p X := L_{S\mathbb{F}_p} X$, $L_{\mathbb{Q}} X := L_{S\mathbb{Q}} X$ (which is the rationalization of X when X is a CW-complex).

That is, we can get the global information of the spectrum X through p-completion, rationalization and how they are glued together. Also, similar to algebra, the p-completion of the space X can be constructed from the completion of the p-localization of X, i.e., $L_{S\mathbb{Z}_{(p)}}X$. This inspires us to investigate the p-localization of a spectrum X, which turns out to have a nice structure and be computable.

Theorem 2.4 (Chromatic Convergence Theorem)**.** *Suppose that X is a p-local spectrum, i.e., it is the p-localization of some spectrum. Then we have $X \cong holim L_n X$, where $L_n X \cong L_{E_n} X$ and E_n is the Morava E-theory (also called Lubin-Tate theory).*

Theorem 2.5 (Smash Product Theorem)**.** *For any spectrum X, $L_n X \cong X \wedge L_n S$.*

Therefore, we can recover the information of X from $L_n X \cong X \wedge L_n S$. Then we are reduced to compute $L_n S$, which can be decomposed further:

Proposition 2.6. *Let $K(n)$ be the Morava K-theory and X be an arbitrary spectrum.*

$$\begin{array}{ccc} L_n X & \longrightarrow & L_{K(n)}X \\ \downarrow & & \downarrow \\ L_{n-1}X & \longrightarrow & L_{n-1}L_{K(n)}X \end{array}$$

Therefore, we are reduced to calculate $L_{K(n)}X$, which is somehow related to the equivariant stuff:

Theorem 2.7 (Devinatz-Hopkins)**.** $L_{K(n)}S \cong E_n^{h\mathbb{G}_n}$, *where $\mathbb{G}_n$ is called the Morava stabilizer group.*

3 Periodicity Theorem

Due to the discussion in Section 2, from now on we will focus on the case of p-local spectra. There is a sequence of useful homology theories in investigating p-local spectra called the Morava K-theory. They will give a filtration of the category of p-local spectra. The construction is tedious and artificial, so we only display some properties of $K(n)$ here:

Proposition 3.1. *For each prime p there is a sequence of homology theories $K(n)_*$ for $n \geqslant 0$ with the following properties.*

(i) $K(0)_*(X) = H_*(X;\mathbb{Q})$ *and* $\overline{K(0)}_*(X) = 0$ *when* $\overline{H}_*(X)$ *is all torsion.*

(ii) *$K(1)_*(X)$ is one of $p-1$ isomorphic summands of mod p complex K-theory.*

(iii) *$K(0)_* = \mathbb{Q}$ and for $n > 0$, $K(n)_* = \mathbb{F}_p[v_n^{\pm}]$ where the dimension of v_n is $2p^n - 2$. This ring is a graded field in the sense that every graded module over it is free. For each $n \geqslant 0$, $K(n)_*(X)$ is a module over $K(n)_*$.*

(iv) *$K(n)_*(X \times Y) \cong K(n)_*(X) \otimes_{K(n)_*} K(n)_*(Y)$.*

(v) *Let X be a p-local finite CW-complex. If $\overline{K(n)}_*(X) = 0$, then $\overline{K(n-1)}_*(X) = 0$.*

(vi) *Let X be a p-local finite CW-complex.*

$$\overline{K(n)}_*(X) = K(n)_* \otimes_{\mathbb{F}_p} \overline{H}_*(X; \mathbb{F}_p)$$

for n sufficiently large. In particular, it is nontrivial if X is simply connected and not contractible.

Definition 3.2 (Type)**.** A p-local finite complex X has **type** n if n is the smallest integer such that $\overline{K(n)}_*(X)$ is nontrivial. If X is contractible, it has **type** ∞.

Besides the types, Morava K-theories are useful in detecting periodic self-maps of a spectrum, which will give finer structures of the homotopy groups.

Theorem 3.3 (Periodicity theorem)**.** *Let X, Y be p-local finite CW-complexes of type n for finite n.*

(i) *There is a map $f\colon \Sigma^{d+i}X \to \Sigma^i X$ for some $i \geqslant 0$ such that $K(n)_*(f)$ is an isomorphism and $K(m)_*(f) = 0$ for $m \neq n$. (We will refer to such a map as a v_n-**map**) When $n = 0$ then $d = 0$, and when $n > 0$ then d is a multiple of $2p^n - 2$.*

(ii) *Suppose $h\colon X \to Y$ is a continuous map. Assume that both have been suspended enough times to be the target of a v_n-map. Let $g\colon \Sigma^e Y \to Y$ be a self-map as in (i). Then there are positive integers i and j with $di = ej$ such that the following diagram commutes up to homotopy.*

$$\begin{array}{ccc} \Sigma^{di}X & \xrightarrow{\Sigma^{di}h} & \Sigma^{di}Y \\ {\scriptstyle f^i}\downarrow & & \downarrow{\scriptstyle g^j} \\ X & \xrightarrow[h]{} & Y \end{array}$$

If we take $h = Id_X$, then the second part of the periodicity theorem says that the v_n-map is unique up to powers.

4 Geometric Chromatic Filtration and Telescope Conjecture

Theorem 2.4 tells us that there is a filtration of the homotopy group:

$$\mathscr{C}_0^a(X) := \pi_*(X)$$
$$\mathscr{C}_n^a(X) := \ker\big(\pi_*(X) \to \pi_*(L_n X)\big)$$
$$\mathscr{C}_0^a(X) \supset \mathscr{C}_1^a(X) \supset \mathscr{C}_2^a(X) \supset \cdots$$

This is called the algebraic chromatic filtration, but $K(n)$ and E_n are so manufactured. In this section, we are aiming to give a geometric model for this filtration.

Lemma 4.1. *Suppose X has type n. Then the cofiber W of the map $f\colon \Sigma^{d+i}X \to \Sigma^i X$ given by Theorem 3.3 has type $n+1$.*

Proof. For each m, we have a long exact sequence:

$$\cdots \to \overline{K(m)}_t(\Sigma^{d+i}X) \xrightarrow{f_*} \overline{K(m)}_t(\Sigma^i X) \to \overline{K(m)}_t(W) \to \overline{K(m)}_{t-1}(\Sigma^{d+i}X) \xrightarrow{f_*} \cdots$$

When $m < n$, $\overline{K(m)}_*(\Sigma^{d+i}X) = \overline{K(m)}_*(\Sigma^d X) = 0$, so $\overline{K(m)}_*(W) = 0$. When $m = n$, f_* are isomorphisms, so $\overline{K(m)}_*(W) = 0$ again. When $m = n+1$, $f_* = 0$, so $\overline{K(m)}_*(W) = \overline{K(m)}_{*-1}(\Sigma^{d+i}X)$ is nontrivial by Proposition 3.1. □

Proposition 4.2. *Let X be a CW-complex and $X_{(p)} := L_{S\mathbb{Z}_{(p)}}X$. Then $\overline{E}_*(X_{(p)}) \cong \overline{E}_*(X) \otimes \mathbb{Z}_{(p)}$. If X is finite, $X_{(p)}$ is also finite.*

Suppose X is a p-local complex. Then each element $x \in \pi_k^S(X)$ has infinite order or order p^i for some i. If y has infinite order, then it has a nontrivial image in $\pi_k^S(X) \otimes \mathbb{Q}$, which is left for rational homotopy theory.

On the other hand, if x has order p^i for some i, then the composite (Here we omit the suspension for simplicity)

$$S^k \xrightarrow{p^i} S^k \xrightarrow{x} X$$

is null-homotopic. Technically, we localize at p here. Then x factors through the cofiber $W(1)$ of $p^i\colon S^k_{(p)} \to S^k_{(p)}$. Note that the sphere spectrum has type 0 and the map p^i is a v_0-map. Thus, $W(1)$ has type 1. Therefore, it admits a v_1-map $f_1\colon \Sigma^{d_1}W(1) \to W(1)$. Hence, we have the following diagram:

$$
\begin{array}{ccccc}
S^k_{(p)} & \xrightarrow{p^i} & S^k_{(p)} & \xrightarrow{y} & X \\
 & & \downarrow & \overset{g_1}{\nearrow} & \\
\Sigma^{d_1}W(1) & \xrightarrow{f_1} & W(1) & &
\end{array}
$$

If the composite of g_1 and all powers of f_1 are not null-homotopic, then g_1 has a nontrivial image in $v_1^{-1}[W(1), Y]^S_*$, which is the colimit

$$[W(1), X]^S_* \xrightarrow{f_1^*} [\Sigma^{d_1}W(1), X]^S_* \xrightarrow{f_1^*} [\Sigma^{2d_1}W(1), X]^S_* \xrightarrow{f_1^*} \cdots$$

On the other hand, if $g_1 f_1^{i_1}$ is null-homotopic for some i_1. Let $W(2)$ be the cofiber of the map $f_1^{i_1} \colon \Sigma^{d_1 i_1}W(1) \to W(1)$. Iterating this process we get a diagram:

$$
\begin{array}{ccccc}
S^k_{(p)} & \xrightarrow{p^i} & S^k_{(p)} & \xrightarrow{x} & X \\
 & & \downarrow & & \\
\Sigma^{d_1 i_1}W(1) & \xrightarrow{f_1^{i_1}} & W(1) & \overset{g_1}{\dashrightarrow} & X \\
 & & \downarrow & & \\
\Sigma^{d_2 i_2}W(2) & \xrightarrow{f_2^{i_2}} & W(2) & \overset{g_2}{\dashrightarrow} & X \\
 & & \downarrow & & \\
 & & \vdots & &
\end{array}
$$

Definition 4.3 (Geometric chromatic filtration)**.** If an element $x \in \pi^S_*(X)$ extends to a p-local complex $W(n)$ of type n, then x is **v_{n-1}-torsion**. If in addition x does not extend to a p-local complex of type $n+1$, it is **v_n-periodic**. The **geometric chromatic filtration** of $\pi^S_*(X)$ is the decreasing family of subgroups consisting of the v_n-torsion elements for various $n \geqslant 0$.

Conjecture 4.4 (Telescope Conjecture)**.** *The algebraic chromatic filtration is the same with the geometric chromatic filtration.*

Finally, we want to talk about why this conjecture is called ”telescope” and interpret the geometric filtration in the viewpoint of Bousfield localization.

Definition 4.5 (Telescope of a self-map)**.** Let $f \colon \Sigma^d X \to X$ be a self-map. Then the **telescope of** f is the homotopy colimit

$$\hat{X} := f^{-1}X := \text{hocolim}\big(X \xrightarrow{\Sigma^{-d}f} \Sigma^{-d}X \xrightarrow{\Sigma^{-2d}f} \Sigma^{-2d}X \xrightarrow{\Sigma^{-3d}f} \cdots\big)$$

By Theorem 3.3(ii), $\hat{X}$ is independent of the choice of f, since the v_n-maps are unique up to powers.

In analogy with algebra, this likes

$$M[f^{-1}] = \text{colim}(M \xrightarrow{f} M \xrightarrow{f} M \to \cdots)$$

where M is an R-module and $0 \neq f \in R$.

Definition 4.6 (Telescope Localization)**.** Let $Tel(n) := f_n^{-1}W(n)$, where $W(n)$ is defined as above. Define the **telescope localization** by

$$L_n^f X := L_{Tel(0)\vee\cdots\vee Tel(n)} X$$

By Theorem 6.6, this definition does not rely on the choice of $W(n)$. Actually, we can take arbitrary p-local finite CW-complex of type n. That is why we take the p-localization at the beginning of the construction.

Example 4.7. If X is of type n with v_n-self map f, then $L_n^f X \cong \hat{X}$. See [Lur10, Lecture 28, Proposition 1]. That is why this is called the "telescope" localization.

Now suppose $x \in \pi_k(X)$ is v_0-torsion, i.e., it can factor through $W(1)$ defined above. Due to 6.5, $\hat{S}^k_{(p)} \wedge W(1)$ is contractible. Therefore, $W(1)$ is $\hat{S}^k_{(p)}$-acyclic. Since $L_0^f X = L_{\hat{S}^k_{(p)}} X$ is $\hat{S}^k_{(p)}$-local, $[W(1), L_{\hat{S}^k_{(p)}} X] = 0$. Hence, x has trivial image in $\pi_*(X) \to \pi_*(L_n^f X)$. Conversely, $Tel(0) = p^{-1}S_{(p)}$, so $H_*(p^{-1}S_{(p)}) = p^{-1}\mathbb{Z}_{(p)} = \mathbb{Q}$. Therefore, $Tel(0) = S\mathbb{Q} = H\mathbb{Q} = K(0)$. If x has trivial image in $\pi_*(X) \to \pi_*(L_0^f X)$, then it factors through the fiber of $X \to L_0^f X$. Since the fiber is $Tel(0)$-acyclic, it has type $\geqslant 1$, so x is v_0-torsion.

This is true for the general case with more knowledge about Bousfield localization. Therefore, under the viewpoint of localization, the geometric chromatic filtration becomes

$$\mathscr{C}_0^g(X) := \pi_*(X)$$
$$\mathscr{C}_n^g(X) := \ker\big(\pi_*(X) \to \pi_*(L_n^f X)\big)$$
$$\mathscr{C}_0^g(X) \supset \mathscr{C}_1^g(X) \supset \mathscr{C}_2^g(X) \supset \cdots$$

And the telescope conjecture says that $\mathscr{C}_n^g = \mathscr{C}_n^a$ or $L_n^f = L_n$ in other word. By above discussion, this is true when $n = 0$. When $n = 1$, the case of $p > 2$ is proved by Miller and the case of $p = 2$ is proved by Mahowald [Bea19, Part III].

The geometric side is more natural and conceptual while the algebraic side is more manufactured and computable. For example, we do not have a chromatic convergence theorem for $L_n^f X$ and the Adams-Novikov spectral sequence may not converge for $\pi_*(L_n^f X)$, so $\pi_*(L_n^f X)$ is hard to compute.

Now suppose that $x\colon S^k \to X$ is v_n-periodic and that it extends to $g_n\colon W(n) \to X$. Suppose $e\colon S^K \to W(n)$ is the bottom cell in $W(n)$. Then for each i, we have a composition

$$S^{K+d_n i} \xrightarrow{\Sigma^{d_n i} e} \Sigma^{d_n i} W(n) \xrightarrow{f_n^i} W(n) \xrightarrow{g_n} X$$

We can play the same game as above to get a nontrivial element in $\pi_*^S(X)$.

Definition 4.8 (v_n-periodic family)**.** Given a v_n-periodic element $x \in \pi_*^S(X)$, the element described above for various $i > 0$ constitute the v_n-periodic family associated with x.

5 Thick Category Theorem

5.1 The category CΓ

Let $L \cong \mathbb{Z}[x_1, x_2, \cdots]$ be the Lazard ring and $G(x, y)$ be the universal formal group law over L.

Definition 5.1. Let Γ be the group of power series over $\mathbb{Z}$ having the form $\gamma = x + b_1 x + b_2 x + \cdots$ where $b_1, b_2, \cdots \in \mathbb{Z}$. Then Γ acts on L by the following. Note that $\gamma^{-1}\Big(G\big(\gamma(x), \gamma(y)\big)\Big) \in FGL(L)$. It is determined by a homomorphism $L \to L$. Since γ is invertible, this endomorphism is an automorphism, which is the desired action.

Let MU be the complex cobordism theory. Then Γ also acts naturally on $MU_*(X)$ compatibly with the action on MU_*.

Remark. *According to [Rav92, Section 3.3], this action is an analogy to the action of the group of multiplicative cohomology operations. For example, in the mod 2 case, we consider* *multiplicative cohomology operations* ϕ

$$\phi\colon H^1(X;\mathbb{F}_2) \to H^*(X;\mathbb{F}_2)[t] = H^*(X \times \mathbb{RP}^\infty;\mathbb{F}_2)$$
$$x \mapsto \sum_{i\geqslant 1}^{\infty} \phi_i(x)$$

Suppose $H^1(\mathbb{RP}^\infty;\mathbb{F}_2) = \mathbb{F}_2[a]$. *For any* $x \in H^1(X;\mathbb{F}_2)$, x *can be viewed as a map* $X \to K(\mathbb{F}_2,1) = \mathbb{RP}^\infty$ *such that the following diagram commutes:*

$$\begin{array}{ccc} H^*(\mathbb{RP}^\infty;\mathbb{F}_2) & \xrightarrow{\phi} & H^*(\mathbb{RP}^\infty;\mathbb{F}_2) \\ {\scriptstyle x^*}\downarrow & & \downarrow{\scriptstyle (x\times id)^*} \\ H^*(X;\mathbb{F}_2) & \xrightarrow[\phi]{} & H^*(X;F_2) \end{array} \qquad \begin{array}{ccc} a & \longmapsto & \sum \phi_i(a) \\ \downarrow & & \downarrow \\ x & \longmapsto & \sum \phi_i(x) \end{array}$$

Therefore, ϕ *is determined by its effect on the generator of* a. *If* ϕ *is in the group of multiplicative cohomology operations,* $\phi_1(a) = t$, *so the group of such multiplicative cohomology operations can be embedded into* $\Gamma_{\mathbb{F}_2}$, *which is the analog of* Γ *over* $\mathbb{F}_2$.

Definition 5.2. Let $\mathbf{C\Gamma}$ be the category of finitely presented graded L-modules equipped with an action of Γ compatible with its action on L. Let **FH** be the homotopy category of finite CW-complexes.

Therefore, MU_* is a functor from **FH** to $\mathbf{C\Gamma}$, which is more accessible and is the main object in this subsection.

Let $v_n \in L$ denote the coefficient of x^{p^n} in the p-series for the universal formal group law. It can be shown that v_n can serve as a polynomial generator in dimension $2p^n - 2$ [Lur10, Lecture 13, Proposition 1]. Let $I_{p,n} := (p, v_1, \cdots, v_{n-1}) \subset L$.

Theorem 5.3 (Invariant Prime Ideal Theorem)**.** *The only prime ideals in* L *which are invariant under the action of* Γ *are the* $I_{p,n}$ *defined above, where* p *is a prime integer and* $n \in \mathbb{N}$, *possibly* ∞. *By convention,* $I_{p,0} = 0$.

Moreover, $(L/I_{p,n})^\Gamma = \mathbb{F}_p[v_n]$ *for* $n > 0$ *and* $L^\Gamma = \mathbb{Z}$.

Proof. For references, see [Rav92, Theorem 3.3.6]. □

Theorem 5.4 (Landweber Filtration Theorem)**.** *Every module* M *in* $\mathbf{C\Gamma}$ *admits a finite filtration by submodules in* $\mathbf{C\Gamma}$ *as above in which each subquotient is isomorphic to a suspension (recall these modules are graded) of* $L/I_{p,n}$ *for some prime* p *and finite* n.

Proof. For references, see [Rav92, Theorem 3.3.7]. □

We may consider $L/I_{p,n}$ classifying formal group laws of height greater or equal to n. Then the filtration looks like a filtration of MU such that each subquotient is a suspension of some universal spectra within the category of complex oriented spectra with formal group laws of height $\geqslant n$.

Remark. *In fact, the Landweber exact functor theorem is proved using the above two theorems.*

Remark. *A finitely generated module M over a Noetherian ring R has a finite filtration with each subquotient equals to R/I for some prime ideal I. Note that L is not Noetherian, but it is a limit of Noetherian rings, so finitely presented modules over it admits similar filtrations. That is why we define* $\mathbf{C\Gamma}$ *to be the category of such modules.*

Corollary 5.5. *Suppose M is a p-local module in* $\mathbf{C\Gamma}$ *and $x \in M$.*

(a) If x is annihilated by some power of v_n, then it is annihilated by some power of v_{n-1}, so if $v_n^{-1}M = 0$, then $v_{n-1}^{-1}M = 0$.

(b) If x is nonzero, then there is an n so that $v_n^k x \neq 0$ for all k, so if M is nontrivial, then so is $v_n^{-1}M$ for all sufficiently large n.

(c) If $v_{n-1}^{-1}M = 0$, then there is a positive integer d such that multiplication by v_n^d in M commutes with the action of Γ.

(d) Conversely, if $v_{n-1}^{-1}M$ is nontrivial, then there is no positive integer k such that multiplication by v_n^k in M commutes with the action of Γ on x.

Proof. Proofs are similar to 5.9. See [Rav92, Corollary 3.3.9]. □

The first two statements are similar to the one of Morava K-theory. In fact, for a finite p-local CW-complex X, $v_n^{-1}\overline{MU}_*(X)_{(p)} = 0$ if and only if $\overline{K(n)}_*(X) = 0$. One can replace $K(n)_*$ by $v_n^{-1}MU_{(p)}$ in the statement of the periodicity theorem. The third statement is an analogy of the periodicity theorem.

Definition 5.6. A p-local module M in $\mathbf{C\Gamma}$ has **type** n if n is the smallest integer with $v_n^{-1}M$ nontrivial. A homomorphism $f\colon \Sigma^d M \to M$ in $\mathbf{C\Gamma}$ is a v_n-map if it induces an isomorphism in $v_n^{-1}M$ and the trivial homomorphism in $v_m^{-1}M$ for $m \neq n$.

Corollary 5.7. *If M in $\mathbf{C\Gamma}$ is a p-local module with $v_{n-1}^{-1}M$ nontrivial, then M does admit a v_n-map.*

Proof. Proof is similar to 5.9. See [Rav92, Corollary 3.3.11]. □

5.2 Thick Subcategories

Definition 5.8 (Thick Subcategory)**.** A full subcategory $\mathbf{C}$ of $\mathbf{C\Gamma}$ is **thick** is it satisfies that if

$$0 \to M' \to M \to M'' \to 0$$

is a short exact sequence, then M is in $\mathbf{C}$ if and only if M', M'' are in it.

A full subcategory $\mathbf{F}$ of $\mathbf{FH}$ is **thick** if it satisfies the following axioms:

(a) If

$$X \xrightarrow{f} Y \to C_f$$

is a cofiber sequence in which two of the three spaces are in $\mathbf{F}$, then so is the third.

(b) If $X \vee Y$ is in $\mathbf{F}$ then so are X and Y.

Using Landweber filtration theorem we can prove that

Theorem 5.9. *Let $\mathbf{C}$ be a thick subcategory of $\mathbf{C\Gamma}_{(p)}$, the subcategory of all p-local modules in $\mathbf{C\Gamma}$. Then $\mathbf{C}$ is either all of $\mathbf{C\Gamma}_{(p)}$, or consists of all p-local modules M in $\mathbf{C\Gamma}$ with $v_{n-1}^{-1}M = 0$. We denote the latter category by $\mathbf{C\Gamma}_{p,n}$.*

Proof. There is largest n such that $\mathbf{C\Gamma}_{p,n} \supset \mathbf{C}$ and $\mathbf{C} \not\subseteq \mathbf{C\Gamma}_{p,n+1}$. Then choose $M \in \mathbf{C} - \mathbf{C\Gamma}_{p,n+1}$. Then $v_n^{-1}M \neq 0$ and $v_{n-1}^{-1}M = 0$. Choose a Landweber filtration of M. Then there is a such quotient equals to a suspension of $MU_*/I_{p,k}$ with $v_n^{-1}MU_*/I_{p,k} \neq 0$. Thus, $k \leqslant n$. Since $v_{n-1}^{-1}MU_*/I_{p,k} = 0$, $k = n$. Therefore, $MU_*/I_{p,n} \in \mathbf{C}$. Note that there is an exact sequence

$$0 \to I_{p,m}/I_{p,n} \to MU_*/I_{p,n} \to MU_*/I_{p,m} \to 0$$

for $m \geqslant n$. Since $\mathbf{C}$ is thick, $MU_*/I_{p,m} \in \mathbf{C}$. For all $N \in \mathbf{C\Gamma}_{p,n}$, $v_{n-1}^{-1}N = 0$. Therefore, every subquotient of N is a suspension of $MU_*/I_{p,m}$ for $m \geqslant n$, so $N \in \mathbf{C}$. □

The proof suggests that $\mathbf{C\Gamma}_{p,n}$ actually consists of modules admitting a filtration such that each subquotient is a suspension of $MU_*/I_{p,m}$ for $m \geqslant n$.

In fact, generators other than $v_1, v_2, \cdots$ act freely on any $M \in \mathbf{C\Gamma}$. If we localize L at p and drop out all generators other than $v_1, v_2, \cdots$, we get $\mathbb{Z}_{(p)}[v_1, v_2, \cdots]$. The cohomology theory associated to this ring is called the Brown-Peterson theory BP. Then we get a filtration of BP by prime ideals

$$0 = I_{p,0} \subset I_{p,1} \subset I_{p,1} \subset \cdots$$

If we view MU as $\mathbb{Z}$ and BP as the stalk $\mathbb{Z}_{(p)}$ of $\mathbb{Z}$ at p, the filtration of BP is a filtration of the stalk by heights. Then any $M \in \mathbf{C\Gamma}_{p,n}$ is a composition of "sub-stalks" of BP consisting of elements of height $\geqslant n$.

There is an analogous result about thick subcategories of $\mathbf{FH}_{(p)}$.

Theorem 5.10 (Thick Category Theorem)**.** *Let* $\mathbf{F}$ *be a thick subcategory of* $\mathbf{FH}_{(p)}$*, the category of* p*-local finite CW-complexes. Then* $\mathbf{F}$ *is either all of* $\mathbf{FH}_{(p)}$*, the trivial subcategory or consists of all* p*-local finite CW-complexes* X *with* $\overline{K(n)}_*(X) = 0$*, which is equivalent to say that* $v_{n-1}^{-1}\overline{MU}_*(X) = 0$*. We denote the latter category by* $\mathbf{FH}_{p,n}$*.*

Therefore, we have two sequences of thick subcategories, where $MU_*(\cdot)$ sends one to the other.

$$\mathbf{FH}_{(p)} = \mathbf{FH}_{p,0} \supset \mathbf{FH}_{p,1} \supset \mathbf{FH}_{p,2} \supset \cdots \supset *$$
$$\mathbf{C\Gamma}_{(p)} = \mathbf{C\Gamma}_{p,0} \supset \mathbf{C\Gamma}_{p,1} \supset \mathbf{C\Gamma}_{p,2} \supset \cdots \supset 0$$

Under such a point of view, the geometric chromatic filtration becomes

$$\begin{array}{ccc} \mathbf{FH}_{p,0} \ni & S^k_{(p)} & \longrightarrow X \\ \cup & \downarrow & \\ \mathbf{FH}_{p,1} \ni & W(1) & \\ \cup & \downarrow & \\ \mathbf{FH}_{p,2} \ni & W(2) & \\ & \downarrow & \\ & \vdots & \end{array}$$

A map $x\colon S^k \to X$ is v_{n-1}-torsion if and only if it can be induced from a smaller thick subcategory $\mathbf{FH}_{p,n}$.

6 Bousfield Equivalence

We first display some easy consequences of the definition of Bousfield localization.

Proposition 6.1. *For any homology theory E_*,*

(a) Any inverse limit of E_*-local spectra is E_*-local.

(b) If

$$W \to X \to Y \to \Sigma W$$

is a cofiber sequence and any two of W, X and Y are E_*-local, then so is the third.

(c) If $X \vee Y$ is E_*-local, then so are X and Y.

In particular, (b)(c) say that E_*-local spectra form a thick subcategory.

Lemma 6.2. *If E is a ring spectrum, then $E \wedge X$ is E_*-local for any spectrum X.*

Proof. Suppose W is E_*-acyclic. Suppose $\eta\colon S \to E$ and $m\colon E \wedge E \to E$ are unit and multiplication map of E respectively. For any $f\colon W \to E \wedge X$, we have

$$\begin{array}{ccccc} W & \xrightarrow{f} & E \wedge X & & \\ \downarrow{\scriptstyle \eta\wedge Id_W} & & \downarrow{\scriptstyle \eta\wedge Id_{E\wedge X}} & \searrow^{Id_{E\wedge X}} & \\ E \wedge W & \xrightarrow[Id_E\wedge f]{} & E \wedge E \wedge X & \xrightarrow[m\wedge Id_X]{} & E \wedge X \end{array}$$

Since $E \wedge W$ is contractible, f is null-homotopic. □

From Section 2, we know that chromatic homotopy theory cares about the localization of spectra. It is natural to ask when two spectra induce the same localization.

Definition 6.3 (Bousfield Localization)**.** Two spectra E, F are **Bousfield equivalent** if for each spectrum X, $E\wedge X$ is contractible if and only if $F\wedge X$ is contractible. The Bousfield equivalence class of E is denoted by $\langle E\rangle$.

Say $\langle E\rangle \geqslant \langle F\rangle$ if for each spectrum X, $E \wedge X$ is contractible implies that $F \wedge X$ is contractible. Say $\langle E\rangle > \langle F\rangle$ if $\langle E\rangle \geqslant \langle F\rangle$ and $\langle E\rangle \neq \langle F\rangle$.

Define $\langle E\rangle \wedge \langle F\rangle := \langle E \wedge F\rangle$ and $\langle E\rangle \vee \langle F\rangle := \langle E \vee F\rangle$.

A class $\langle E\rangle$ has a **complement** $\langle E\rangle^c$ if $\langle E\rangle \wedge \langle E\rangle^c = \langle *\rangle$ and $\langle E\rangle \vee \langle E\rangle^c = \langle S\rangle$.

We can show that $\wedge$ and $\vee$ satisfy the distributive laws.

$$(\langle X\rangle \wedge \langle Y\rangle) \vee \langle Z\rangle = (\langle X\rangle \vee \langle Z\rangle) \wedge (\langle Y\rangle \vee \langle Z\rangle)$$
$$(\langle X\rangle \vee \langle Y\rangle) \wedge \langle Z\rangle = (\langle X\rangle \wedge \langle Z\rangle) \vee (\langle Y\rangle \wedge \langle Z\rangle)$$

Proposition 6.4. *The Bousfield localizations $L_E \cong L_F$ if and only if $\langle E\rangle = \langle F\rangle$. If $\langle E\rangle \leqslant \langle F\rangle$ then $L_E L_F = L_E$ and there is a natural transformation $L_F \to L_E$.*

There is another property that is similar to the relation between localizations and quotients $p^{-1}S \otimes S/p = 0$ in commutative algebra.

Proposition 6.5. *Given a self-map $f\colon \Sigma^d X \to X$, let C_f denote its cofiber and let $\hat{X}$ be the telescope. Then $\langle X\rangle = \langle \hat{X}\rangle \vee \langle C_f\rangle$ and $\langle \hat{X}\rangle \wedge \langle C_f\rangle = \langle *\rangle$.*

For any spectrum E,

$$\langle S\rangle \geqslant \langle E\rangle \geqslant \langle *\rangle$$
$$\langle S\rangle \wedge \langle E\rangle = \langle E\rangle$$
$$\langle S\rangle \vee \langle E\rangle = \langle S\rangle$$
$$\langle *\rangle \vee \langle E\rangle = \langle E\rangle$$
$$\langle *\rangle \wedge \langle E\rangle = \langle *\rangle$$

Thus, Bousfield equivalence classes with complements form a Boolean algebra $\mathbf{BA}$. We have a structure theorem for part of this algebra. Firstly, we have a corollary of Thick category theorem.

Theorem 6.6 (Class Invariance Theorem)**.** *Let X, Y be p-local finite CW-complexes of types m, n respectively. Then $\langle X\rangle = \langle Y\rangle$ if and only if $m = n$ and $\langle X\rangle < \langle Y\rangle$ if and only if $m > n$.*

Proof. Suppose that C_X, C_Y are the smallest thick subcategories containing X, Y respectively. Then C_X consists of finite complexes built from X through cofibrations and retracts. Therefore, $\langle X'\rangle \leqslant \langle X\rangle$ for all X' in C_X. Since $X \wedge K(m-1) = 0$, $X' \wedge K(m-1) = 0$. Thus, $C_X \subset \mathbf{FH}_{p,m}$. Since $K_*(m)(X) \neq 0$, $C_X \nsubseteq \mathbf{FH}_{p,m+1}$. Therefore, $C_X = \mathbf{FH}_{p,m}$. Similarly, $C_Y = \mathbf{FH}_{p,n}$. Then $C_X = C_Y$ if and only if $m = n$. If $C_X = C_Y$, then $\langle X\rangle \leqslant \langle Y\rangle$ and $\langle X\rangle \geqslant \langle Y\rangle$. Thus, $C_X = C_Y$ if and only if $\langle X\rangle = \langle Y\rangle$.

The inequality can be proved similarly. ☐

Pick a p-local CW-complex X_n of type n, the Bousfield equivalence class $\langle X_n \rangle$ and the telescope $\langle \hat{X}_n \rangle$ are independent of the choice of X. The following theorem gives a description of the structure of part of $\mathbf{BA}$.

Theorem 6.7 (Boolean Algebra Theorem)**.** *Let* $\mathbf{FBA} \subset \mathbf{BA}$ *be the Boolean subalgebra generated by finite spectra and their complements. Let* $\mathbf{FBA}_{(p)} \subset \mathbf{FBA}$ *be the subalgebra of* p*-local finite spectra and their complements in* $\langle S_{(p)} \rangle$*. Then* $\mathbf{FBA}_{(p)}$ *is the free (under complements, finite unions and finite intersections) Boolean algebra generated by the classes of the telescopes* $\langle \hat{X}_n \rangle$ *for* $n \geqslant 0$*.*

Proof. See [Rav92, Theorem 7.2.9]. □

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